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AN AXIOMATIC MODEL OF RATE-INDEPENDENT PLASTICITY

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Abstract—A model of rate-independent plasticity is proposed, in which neither loading surfaces nor yield surfaces are assumed to exist. It is shown that the existence of loading surfaces follows mathematically from a loading/unloading postulate and the continuity of material behavior and that the second law of thermodynamics can furnish a yield criterion which may be identified with the Frank–Read source activation criterion.

1. INTRODUCTION

In the course of its development in the present century, the theory of plasticity has become enmeshed in a tangled web of interdependent postulates: yield criteria, hardening rules, normality and stability principles, quasi-thermodynamic postulates on work cycles and others. A few years ago I presented [1, 2] a tentative proposal to simplify the structure of plasticity theory by basing it on a minimal definition of plasticity, namely the loading-unloading irreversibility: a material element in a plastic state behaves plastically upon loading and elastically upon unloading. It is my purpose in this brief note (brief on account of the simplicity of the ideas) to construct a formal axiomatic model of rate-independent plasticity. By "axiomatic" I mean that some unproved assumptions are formally presented, with the remaining properties of the model following from them.

The essential results to be presented are, first, that the existence of loading surfaces in temperature-deformation space follows from the defining (loading/unloading) postulate and from an assumption on the continuity of material behavior; and, second, that a yield criterion (in the strict sense of a criterion for the beginning of plastic behavior)—through it need not exist—may be derived from the second law of thermodynamics.

By a material element I mean a body so small that (a) its deformation (from a reference configuration) is determined solely by the right Cauchy–Green tensor and (b) heat flow within the element can be neglected. The thermomechanical state of the element is thus specified by the state vector $\mathbf{s} = (\Lambda, \mathbf{q})$ in the state space $\mathbf{R}^7 \times \mathbf{Q}$. Here Λ stands for $(\theta, \mathbf{C})$, θ being the absolute temperature and $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ being the right Cauchy–Green tensor, where $\mathbf{F}$ is the displacement gradient; and $\mathbf{q} \in \mathbf{Q}$ is a finite-dimensional material vector (that is, one which is unaffected by a superposed rigid-body motion) whose components are the instantaneous values of the internal variables.

In accordance with current models of plasticity, the displacement gradient $\mathbf{F}$ is decomposed as

$$\mathbf{F} = \mathbf{F}_e\mathbf{F}_p,$$

where, if $\mathrm{d}X$ is an infinitesimal vector in a stress-free reference configuration of the element, then $\mathbf{F}_p\,\mathrm{d}X$ is the image of $\mathrm{d}X$ in the likewise stress-free but plastically deformed "intermediate" configuration. The intermediate configuration is chosen in such a way that $\mathbf{F}_p$ is a material tensor (it is then called "isoclinic" [3]). It is important to note that the entire tensor $\mathbf{F}_p$, and not merely its right Cauchy–Green tensor $\mathbf{C}_p = \mathbf{F}_p{}^T\mathbf{F}_p$, must in general be included in a description of the state, except in certain cases of isotropy. Consequently, the components of $\mathbf{F}_p$ must be included among those of $\mathbf{q}$, so that the dimension of $\mathbf{Q}$ is at least nine.

The essential difference between the rate-dependent (or viscoplastic) and rate-independent models of plasticity is that in the former the material will respond elastically in a sufficiently fast process; that is, if the temperature-deformation rate vector $\dot{\Lambda}$ is sufficiently large in magnitude, the change in $\mathbf{q}$ will be negligible. In rate-independent plasticity, on the other hand, there are directions of $\dot{\Lambda}$ such that q must change (with plastic deformation taking place) regardless of the magnitude of $\dot{\Lambda}$. Such directions will be called plastic directions, and the remaining directions will be called elastic, as in the following formal definition.

2. PLASTICITY AND LOADING SURFACES

Definitions

(1) A vector $\Gamma \in \mathbf{R}^7$ is an elastic (plastic) direction at $s = (\Lambda, q)$ if there exists a $k > 0$ such that $s' = (\Lambda + h\Gamma, q)$ is attainable (unattainable) from s for any $h \in (0, k)$. The set of elastic directions at s will be denoted $A(s)$ and its closure $\bar{A}(s)$; the complement of $\bar{A}(s)$ will be denoted $B(s)$.

(2) s is a plastic state if there exists a plastic direction at s.

(3) s is a regular plastic state if $\Gamma \in B(s)$ implies $-\Gamma \in \bar{A}(s)$; the set of regular plastic states will be denoted P.

Axiom 1. There exists an open set P' of plastic states such that P is a dense subset of P', and such that the following postulates hold at all $s \in P'$: (a) If $s' = (\Lambda', q)$ and $s'' = (\Lambda'', q)$ belong to a neighborhood of $s = (\Lambda, q)$ in P', such that s′ is attainable from s and s″ is attainable from s′, then s″ is attainable from s. (b) The mapping $s \mapsto \bar{A}(s)$ is continuous. (For defining continuity, we may use a metric d' on a collection C of closed sets in $\mathbf{R}^n$, defined as follows: $d'(A, B) = \sup_{x \in A} d(x, B) + \sup_{x \in B} d(x, A)$, $A \in C$, $B \in C$, d being a standard metric on $\mathbf{R}^n$).

Lemma 1. At every $s \in P'$, $\bar{A}(s)$ is a closed convex cone.

Proof: Let $\Lambda' = \Lambda + h\Gamma$, with $\Gamma \in A(s)$, $0 < h < k$, and $\Lambda'' = \Lambda + h'\Gamma'$, with $\Delta' \in A(s')$, $0 < h' < k'$; then, by postulate 1(a), $h\Gamma + h'\Delta' \in A(s)$ for any $h \in (0, k)$ and any $h' \in (0, k')$, or, equivalently, $a\Gamma + b\Delta' \in A(s)$ for any $a, b > 0$. But from postulate 1(b) we infer that if $s' = s + O(h)$ then $\lim_{h \to 0} \sup_{\Delta \in A(s)} \inf_{\Delta' \in A(s')} |\Delta' - \Delta| = 0$; thus for any $\Delta \in \bar{A}(s)$, $\lim_{h \to 0} \inf_{\Delta' \in A(s')} |\Delta' - \Delta| = 0$, so that for every $\Delta \in \bar{A}(s)$ there exists a $\Delta' \in \bar{A}(s')$ such that $\lim_{h \to 0} \Delta' = \Delta$. Consequently $a\Gamma + b\Delta \in \bar{A}(s)$ for every $\Gamma, \Delta \in \bar{A}(s)$ and $a, b > 0$.

Lemma 2. At every $s = (\Lambda, q) \in P$ there exists a direction $\Omega(\Lambda, q)$ such that $\Gamma \in \bar{A}(s)$ if an only if $\Gamma \cdot \Omega(\Lambda, q) \leq 0$.

Proof: By Lemma 1, $B(s)$ is an open concave cone; but to satisfy regularity, it must reduce to an open half-space so that $\bar{A}(s)$ is a closed half-space, its boundary being a hyperplane; the lemma follows if $\Omega(\Lambda, q)$ is a vector normal to this hyperplace and pointing into $B(s)$.

The continuity postulate 1(b) obviously also implies that $\Omega(\cdot, q)$ is continuous at every Λ with $(\Lambda, q) \in P$; this means that $\Omega(\Lambda, q) \cdot d\Lambda$ is a Pfaffian form in $\mathbf{R}^7$. Since states with $\Omega(\Lambda, q) \cdot d\Lambda < 0$ are attainable from (Λ, q), we have a sufficient condition for the operation of Carathéodory's principle, which we can state as a theorem without formal proof.

Theorem. At a state $(\Lambda, q) \in P$ there exists a neighborhood N of Λ and real-valued functions $g(\cdot, q) \in C^1(N)$, $h(\cdot, q) \in C(N)$ such that

$$\Omega(\Lambda, q) = h(\Lambda, q) \frac{\partial}{\partial \Lambda} g(\Lambda, q).$$

In other words, $\Omega(\Lambda, q)$ is normal to the surface (in $\mathbf{R}^7$) $g(\Lambda, q) = \text{constant}$, passing through Λ. Such a surface is called a loading surface and g (which is not unique, since it can be replaced by any continuously differentiable function of g) is called a *loading function*. It is convenient to define the pair (g, h) so that h is positive. A plastic direction at (Λ, q) will also be called a *loading direction*. Specifically, we are interested in the direction represented by $\dot{\Lambda}$. We have a loading direction if

$$\mathring{g} > 0,$$

where

$$\dot{g} \stackrel{\text{def}}{=} \frac{\partial}{\partial \Lambda} g(\Lambda, \mathbf{q}) \cdot \dot{\Lambda}$$

An elastic direction $\dot{\Lambda}$ is called *unloading* if $\dot{g} < 0$ and *neutral loading* if $\dot{g} = 0$.

A *flow rule* is a function $\boldsymbol{\phi}: P \times \mathbf{R}^7 \to \mathbf{Q}$ such that

$$\dot{\mathbf{q}} = \boldsymbol{\phi}(\Lambda, \mathbf{q}; \dot{\Lambda}).$$

We note that $\dot{\mathbf{q}} = 0$ if $\dot{g} \leq 0$, and $\dot{\mathbf{q}} \neq 0$ if $\dot{g} > 0$. The material is rate-independent if $\dot{\mathbf{q}}$ is homogeneous of the first degree in $\dot{\Lambda}$. The simplest rate-independent flow rule is then

$$\dot{\mathbf{q}} = \mathbf{w}(\Lambda, \mathbf{q}) \langle \dot{g} \rangle \tag{1}$$

where w: $P \to \mathbf{Q}$ is assumed continuous and $\langle \cdot \rangle$ denotes the ramp function or Macauley bracket.

More specifically, the term "flow rule" may be reserved for those components of $\mathbf{q}$ which correspond to $\mathbf{F}_p$. If we define the plastic distortion rate $\mathbf{L}_p = \dot{\mathbf{F}}_p \mathbf{F}_p^{-1}$, then we may write

$$\mathbf{L}_p = \mathbf{N}(\Lambda, \mathbf{q}) \langle \dot{g} \rangle. \tag{2}$$

Since g is not unique, neither is $\mathbf{w}$ or $\mathbf{N}$. It may be convenient to have $\mathbf{N}(\Lambda, \mathbf{q})$ as a unit tensor in the sense that $\text{tr}[\mathbf{N}(\Lambda, \mathbf{q})\mathbf{N}^T(\Lambda, \mathbf{q})] = 1$; then $\dot{g}$ (whenever it is positive) may be interpreted as the magnitude of the plastic deformation rate. For example, it is known from dislocation theory that the plastic deformation rate due to a single slip system is given [3] by

$$\mathbf{L}_p = \mathbf{m} \otimes \mathbf{n} |\dot{\gamma}|,$$

where $\dot{\gamma}$ is the shear rate, $\mathbf{m}$ is the unit vector in the slip direction and $\mathbf{n}$ is the unit slip-plane normal (in the intermediate configuration). Then we have $|\dot{\gamma}| = \langle \dot{g} \rangle$, and $\mathbf{N} = \mathbf{m} \otimes \mathbf{n}$.

It follows from Definition 1 that s is not a plastic state if all directions are attainable. It will be assumed that it is then an elastic state, and the set of elastic states will be denoted E. Of course, $E \cap P = \emptyset$. But the common boundary of E and P, $\bar{E} \cap \bar{P}$, is not in general empty. If $\mathring{E} \neq \emptyset$, then this boundary is a manifold called the *yield hypersurface*. Howerer, there exists the limiting case with $\mathring{E} = \emptyset$, when $\bar{E}$ is itself a hypersurface (*quasi-yield hypersurface*) on which $\boldsymbol{\Omega}$ is discontinuous [2]. In this case there are no true elastic states, only limiting states between two plastic domains. This kind of behavior appears to be shown by graphite [4].

3. THERMODYNAMICS AND YIELD CRITERIA

We will now look at the thermodynamics of a rate-independent plastic material element. The free energy (per unit volume in the reference configuration) is assumed to take the form

$$\psi = \bar{\psi}(\theta, \mathbf{C}_e, \mathbf{q}),$$

where $\mathbf{C}_e = \mathbf{F}_e^T \mathbf{F}_e$. We note that $\mathbf{C} = \mathbf{F}_p^T \mathbf{C}_e \mathbf{F}_p$, hence

$$\dot{\mathbf{C}} = \mathbf{F}_p^T [\dot{\mathbf{C}}_e + 2(\mathbf{C}_e \mathbf{L}_p)^S] \mathbf{F}_p$$

and

$$\dot{\mathbf{C}}_e = (\mathbf{F}_p^{-1})^T \dot{\mathbf{C}} \mathbf{F}_p^{-1} - 2(\mathbf{C}_e \mathbf{L}_p)^S,$$

where $(\cdot)^S$ denotes "symmetric part". Therefore, by the chain rule,

$$\dot{\psi} = \frac{\partial \bar{\psi}}{\partial \theta} \dot{\theta} + \text{tr}\left[\mathbf{F}_p^{-1} \frac{\partial \bar{\psi}}{\partial \mathbf{C}_e} (\mathbf{F}_p^{-1})^T \dot{\mathbf{C}}\right] - 2\,\text{tr}\left(\frac{\partial \bar{\psi}}{\partial \mathbf{C}_e} \mathbf{C}_e \mathbf{L}_p\right) + \frac{\partial \bar{\psi}}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}}. \tag{3}$$

The second law of thermodynamics for a material element will be expressed in the form of the Clausius–Planck inequality, namely,

$$\sigma = -\dot{\psi} - \eta\dot{\theta} + \frac{1}{2}\operatorname{tr}(\mathbf{P}\dot{\mathbf{C}}) \geq 0, \tag{4}$$

where η is entropy per unit reference volume, and $\mathbf{P}$ is the symmetric Piola stress tensor. Applying the flow rules (1) and (2) and the expression (3) to (4), one obtains

$$\sigma = -\left(\eta + \frac{\partial\bar{\psi}}{\partial\theta}\right)\dot{\theta} + \operatorname{tr}\left\{\left[\frac{1}{2}\mathbf{P} - \mathbf{F}_p^{-1}\frac{\partial\bar{\psi}}{\partial\mathbf{C}_e}(\mathbf{F}_p^{-1})^T\right]\dot{\mathbf{C}}\right\}\lambda\langle\hat{g}\rangle$$
$$\geq 0, \tag{5}$$

where

$$\lambda = 2\operatorname{tr}\left(\frac{\partial\bar{\psi}}{\partial\mathbf{C}_e}\mathbf{C}_e\mathbf{N}\right) - \frac{\partial\bar{\psi}}{\partial\mathbf{q}}\cdot\mathbf{w}.$$

If one accepts the usual thermodynamic relations

$$\eta = -\frac{\partial\bar{\psi}}{\partial\theta}, \tag{6}$$

$$\mathbf{P} = \mathbf{F}_p^{-1}\frac{\partial\bar{\psi}}{\partial\mathbf{C}_e}(\mathbf{F}_p^{T})^{-1} \tag{7}$$

a priori, then the Clausius–Planck inequality is equivalent to

$$\lambda \geq 0. \tag{8}$$

However, the relations (6) and (7) do *not* follow from (5). Assuming only the validity of (8), it is readily seen that (5) is valid if

$$\eta = -\frac{\partial\bar{\psi}}{\partial\theta} + a\lambda\frac{\partial g}{\partial\theta}$$

and

$$\mathbf{P} = \mathbf{F}_p^{-1}\frac{\partial\bar{\psi}}{\partial\mathbf{C}_e}(\mathbf{F}_p^{T})^{-1} - 2\,a\lambda\frac{\partial g}{\partial\mathbf{C}}$$

for *any* $a \in [0, 1]$. The rate of energy dissipation per unit volume is then

$$\sigma = \lambda(\langle\hat{g}\rangle - a\hat{g}).$$

At this point it is necessary to assert an additional axiom concerning the nature of elastic response. (An elastic process, let us recall, is one in which $\mathbf{q}$ remains constant).

Axiom 2 (elasticity). An elastic process is non-dissipative; that is, $\hat{g} \leq 0$ implies $\sigma = 0$.

Corollary 1. $a = 0$; consequently, the relations (6) and (7) hold, and $\sigma = \lambda\langle\hat{g}\rangle$.

Let us define a new material stress tensor Σ by

$$\Sigma = \mathbf{C}_e\mathbf{F}_p\mathbf{P}\mathbf{F}_p^{T} = (\det\mathbf{F})\mathbf{F}_e^{T}\mathbf{T}(\mathbf{F}_e^{T})^{-1},$$

'where $\mathbf{T}$ is the Cauchy stress tensor. We note that Σ is in general (unless $\mathbf{C}_e = \mathbf{I}$) asymmetric;

however, it differs from $\mathbf{T}$ by quantities of order $|\mathbf{F}_e - \mathbf{I}|$ (since we usually have det $\mathbf{F}_p \doteq 1$). Furthermore, the stress power in plastic deformation (per unit reference volume) is given by $\mathrm{tr}(\boldsymbol{\Sigma}^T \mathbf{L}_p)$.

As a consequence of relation (7) we have the additional result:

Corollary 2. The Clausius–Planck inequality is equivalent to

$$\lambda = \mathrm{tr}\,(\boldsymbol{\Sigma}^T \mathbf{N}) - (\partial\bar{\psi}/\partial \mathbf{q}) \cdot \mathbf{w} \geq 0. \tag{9}$$

If, furthermore, $\mathbf{N}$ is normalized as discussed above, then $\mathrm{tr}(\boldsymbol{\Sigma}^T\mathbf{N})$ may be regarded as the resolved stress in the direction of a virtual plastic deformation $\mathbf{N}\delta\gamma$, while $\mathbf{w}$ may be interpreted as the change in $\mathbf{q}$ corresponding to a unit of plastic deformation. Thus, $(\partial\bar{\psi}/\partial \mathbf{q}) \cdot \mathbf{w}$ is the isothermal change in free energy (per unit reference volume) accompanying a unit of plastic deformation, at constant $\mathbf{C}_e$. Corollary 2 then states that a state cannot be plastic unless the stress is at least equal to this energy change. This condition may, of course, be satisfied identically; but if it is not, then $(\partial\bar{\psi}/\partial \mathbf{q}) \cdot \mathbf{w}$ is a lower bound to the *yield stress*.

In fact, if $(\partial\bar{\psi}/\partial \mathbf{q}) \cdot \mathbf{w}$ is nothing but the change in dislocation line energy corresponding to a unit of plastic deformation, then $\lambda = 0$ corresponds precisely to the derivation of the Frank-Read source activation stress[5], and it is this stress which is usually identified with the initial yield stress; that is, the stress required for the *beginning* of plastic deformation. It is common to use other definitions of yield stress, such as the stress required to produce a *sizable* plastic deformation of some prescribed amount. However, only the initial yield stress is consistent with the notion of yield hypersurface used here in the sense of the boundary between the elastic and plastic domains. And, as we have seen, this boundary (or at least an inner bound to it) may be furnished by thermodynamics alone.

4. ADDITIONAL REMARKS

(1) All the preceding results can be derived by using a control space other than temperature-deformation, e.q. temperature-stress or energy-deformation, with no significant difficulties.

(2) Throughout the work, "elastic" may be replaced by "viscoelastic" if all processes are assumed to be rapid; that is, being of a duration much shorter than the shortest significant relaxation time. Of course, they must not be so rapid that the temperature becomes unmeasurable.

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